Discretization of BSDE with arbitrarily irregular terminal condition

Azmi MAKHLOUF
(Osaka University & JST)

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BSDE

Backward Stochastic Differential Equations (BSDEs)

$$\begin{cases} -dY_t &= f(\omega, t, Y_t, Z_t)dt - Z_t dW_t, \\ Y_T &= \xi. \end{cases} \tag{1}$$

- T: terminal time
- ξ: terminal condition
- f: driver/generator
- (Y, Z): the solution (Y is continuous adapted, and Z is predictible).
- Z: control variable

Assumptions. For some $r > 1$,

(i) $(\mathbf{A}_\xi)$: $\mathbb{E}|\xi|^r < \infty$

(ii) $(\mathbf{A}_f)$: f is uniformly Lipschitz-continuous w.r.t. (y, z):

$$|f(\omega, t, y_2, z_2) - f(\omega, t, y_1, z_1)| \leq L_f(|y_2 - y_1| + |z_2 - z_1|),$$

and $\mathbb{E}[\left(\int_0^T |f(s, 0, 0)|\mathrm{d}s\right)^r] < \infty$.

Under these assumptions, BSDE (1) has a unique solution (Y, Z) such that

$$\mathbb{E}[\sup_{t\in[0,T]} |Y_t|^r] + \mathbb{E}[(\int_0^T |Z_t|^2 dt)^{r/2}] \leq C(\mathbb{E}|\xi|^r + \mathbb{E}[(\int_0^T |f(s, 0, 0)|\mathrm{d}s)^r]).$$

Markovian BSDE

Forward component

$$\begin{cases} X_0 & = x_0, \\ dX_t & = b(t, X_t)dt + \sigma(t, X_t)dW_t, \end{cases}$$

$X_t \in \mathbb{R}^d, \quad W_t \in \mathbb{R}^q.$

Backward component

$$\begin{cases} -dY_t & = f(t, X_t, Y_t, Z_t)dt - Z_t dW_t, \\ Y_T & = g(X_T). \end{cases}$$

$Y_t \in \mathbb{R}, \ Z_t \in \mathbb{R}^{1\times q}.$

Link with PDEs

$\mathcal{L}_X :=$ infinitesimal generator of X. If u is the smooth solution to

$$\partial_t u(t,x) + \mathcal{L}_X u(t,x) + f(t,x,u(t,x),\nabla u\sigma(t,x)) = 0, t < T,$$
$$u(T,x) = g(x),$$

then

$$Y_t = u(t, X_t),$$
$$Z_t = \nabla u\sigma(t, X_t).$$

In general, there is no explicit solution.
Usual approximation: "Dynamic Programming Equation":
$Y_{t_N}^N = g(X_T^N)$; for $n = N-1 \dots 0$,

$$\begin{cases} Y_{t_n}^N &= \mathbb{E}(Y_{t_{n+1}}^N + (t_{n+1}-t_n) f(t_n, X_{t_n}^N, Y_{t_{n+1}}^N, Z_{t_n}^N) | \mathcal{F}_{t_n}), \\ Z_{t_n}^N &= \frac{1}{(t_{n+1}-t_n)} \mathbb{E}(Y_{t_{n+1}}^N (W_{t_{n+1}} - W_{t_n})^* | \mathcal{F}_{t_n}), \end{cases}$$

But: the analysis of convergence requires some regularity on the terminal function g.

Time Discretization

Set $h := \frac{T}{N}$ and $t_n := nh$, for $n = 0 \dots N$.
$(Y_t^p, Z_t^p)_{p=0\dots P}$: Picard approximation of BSDE (1), performed on each time interval $[t_n, t_{n+1})$.
We denote by $(Y_.^{p,n}, Z_.^{p,n})$ the restriction of $(Y_.^p, Z_.^p)$ to $[t_n, t_{n+1}), n = N-1 \dots 0$.
We set

$$(Y_{t_N}^{P,N}, Z_{t_N}^{P,N}) := (\xi, 0),$$

then, going backward in time,

$$\left\{\begin{array}{lcl} Y_t^{0,n-1} & = & Y_{t_n}^{P,n} - \int_t^{t_n} Z_s^{0,n-1} \mathrm{d}W_s, \\ Y_t^{1,n-1} & = & Y_{t_n}^{P,n} + \int_t^{t_n} f(s, Y_s^{0,n-1}, Z_s^{0,n-1}) \mathrm{d}s - \int_t^{t_n} Z_s^{1,n-1} \mathrm{d}W_s, \\ & \vdots & \\ Y_t^{P,n-1} & = & Y_{t_n}^{P,n} + \int_t^{t_n} f(s, Y_s^{P-1,n-1}, Z_s^{P-1,n-1}) \mathrm{d}s - \int_t^{t_n} Z_s^{P,n-1} \mathrm{d}W \end{array}\right.$$

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Stability. For this lemma, we let the driver also depend on the Picard iteration.

Lemma

$$\mathbb{E}[\sup_{t\in[0,T]}|Y_t^p|^2]+\mathbb{E}[\int_0^T|Z_s^p|^2\mathrm{d}s]$$

$$\leq C_P\{\mathbb{E}|\xi|^2+\sum_{i=0}^P h^{P-i}\mathbb{E}[(\int_0^T|f^i(s,0,0)|\mathrm{d}s)^2]$$

$$+\sum_{i=0}^p h^{p-i}\mathbb{E}[(\int_0^T|f^i(s,0,0)|\mathrm{d}s)^2]\}.$$

Time discretization error between (Y, Z), solution to BSDE (1), and its discretization (Y^p, Z^p).

Proposition

$$\mathbb{E}[\sup_{t\in[0,T]} |Y_t^p - Y_t|^2] + \mathbb{E}[\int_0^T |Z_s^p - Z_s|^2 \mathrm{d}s]$$
$$\leq C_P h^p(\mathbb{E}[|\xi|^2] + \mathbb{E}[(\int_0^T |f(s,0,0)|\mathrm{d}s)^2]).$$

In particular, $|Y_0^P - Y_0| \leq C_P h^{\frac{P}{2}}$.

Propagation of the regularity in the Markovian case

General notations.

(i) The kth derivative of a function $\varphi(x^1,\dots,x^d)$ w.r.t. $x^{\alpha_1},\dots,x^{\alpha_k}$ (with $\alpha_1,\dots,\alpha_k\in\{1,\dots,d\}$) is denoted by $\partial^k_{x^{\alpha_1},\dots,x^{\alpha_k}}\varphi(x)$, and $\partial^k\varphi(x)$ denotes the matrix of all the kth derivatives at x.

(ii) $|\varphi|_\infty := \sup_{x\in\mathbb{R}^d}|\varphi(x)|$.
For $k\geq 1$, $|\partial^k\varphi|_\infty := \sup_{\alpha_1,\dots,\alpha_k\in\{1,\dots,d\}}|\partial^k_{x^{\alpha_1},\dots,x^{\alpha_k}}\varphi|_\infty$.
$|\varphi|_{k,\infty} := \sum_{l=0}^k|\partial^l\varphi|_\infty$ (and $|\varphi|_{0,\infty} := |\varphi|_\infty$).

(iii) $\mathcal{C}^k_b$ is the set of functions φ s.t. $|\varphi|_{k,\infty}<\infty$ ($\mathcal{C}^0_b$ is the set of bounded measurable functions).

Markovian BSDE

$$\begin{cases} X_0 &= x_0, \\ dX_t &= b(t, X_t)dt + \sigma(t, X_t)dW_t, \end{cases}$$

$$\begin{cases} -dY_t &= f(t, X_t, Y_t, Z_t)dt - Z_t dW_t, \\ Y_T &= g(X_T). \end{cases}$$

$b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times q}$,
$f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{1\times q} \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$.

Assumptions.

(i) $(\mathbf{A}^k_{b,\sigma})$ (for some $0 \leq k \leq \infty$): b and σ belong to $\mathcal{C}^k_b$ w.r.t. the space variable, with γ-Hölder continuous k^{th} derivative, for some $\gamma \in (0,1]$. b and σ are also $\frac{1}{2}$-Hölder continuous w.r.t. the time variable. In addition, σ is uniformly elliptic, i.e. there exists $\delta > 0$ such that $\sigma\sigma^* \geq \delta Id$.

(ii) $(\mathbf{A}^k_f)$ (for some $0 \leq k \leq \infty$): f has uniform linear growth w.r.t. (y,z): $|f(t,x,y,z)| \leq C^0_f(1+|y|+|z|)$. And, if $k \geq 1$, f is k times differentiable (w.r.t. all variables x, y, z) with uniformly bounded derivatives.

(iii) $(\mathbf{A}^0_g)$: g is a bounded measurable function.

Deterministic version: We will show later that $Y_t^p = u^p(t, X_t)$ and $Z_t^p = \partial u^p(t, X_t)\sigma(t, X_t)$, where u^p is defined, for $n = N-1 \dots 0$ and $p = 0 \dots P-1$, by

$$\begin{cases} u^{P,N}(t_N, x) & = & g(x); \\ u^{0,n}(t, x) & = & \mathbb{E}(u^{P,n+1}(t_{n+1}, X_{t_{n+1}}^{t,x})); \\ u^{p+1,n}(t, x) & = & u^{0,n}(t, x) \\ & & +\mathbb{E}\int_t^{t_{n+1}} f(s, X_s^{t,x}, u^{p,n}(s, X_s^{t,x}), \partial u^{p,n}\sigma(s, X_s^{t,x}))\mathrm{d}s. \end{cases}$$

Goal: regularity and estimates for u^p.

Motivation: numerical approximation.

$$\begin{cases} \overline{u}^{0,n}(t^{n,i},x^i) & := \frac{1}{M}\sum_{m=1}^{M} \overline{u}^{P,n+1}(t_{n+1},\overline{X}_{t_{n+1}}^{t^{n,i},x^i,m}); \\ \overline{u}^{0,n}(t,x) & := \mathcal{P}^n \overline{u}^{0,n}(t,x); \\ & \\ \overline{u}^{p+1,n}(t^{n,i},x^i) & := \overline{u}^{0,n}(t^{n,i},x^i) \\ & + \frac{t_{n+1}-t^{n,i}}{M}\sum_{m=1}^{M} f\Big\{\tau^{p,n,i,m},\overline{X}_{\tau^{p,n,i,m}}^{t^{n,i},x^i,m}, \\ & \quad \overline{u}^{p,n}(\tau^{p,n,i,m},\overline{X}_{\tau^{p,n,i,m}}^{t^{n,i},x^i,m}), \\ & \quad \partial\overline{u}^{p,n}\sigma(\tau^{p,n,i,m},\overline{X}_{\tau^{p,n,i,m}}^{t^{n,i},x^i,m})\Big\}; \\ \overline{u}^{p+1,n}(t,x) & := \mathcal{P}^n\overline{u}^{p+1,n}(t,x); \end{cases}$$

$\longrightarrow$ three errors: Euler, Monte-Carlo and linear approximation ($\mathcal{P}$).

- Euler error: $\frac{C(P,N,\mathcal{P})}{\sqrt{N^X}}(1+|u^P|_\infty+|\partial u^P|_\infty)$.
- Monte-Carlo error: $\frac{C(P,N,N^X,\mathcal{P})}{\sqrt{M}}(1+|u^P|_\infty+|\partial u^P|_\infty)$.
- linear approximation error: $C(P,N,\mathcal{P})(|\mathcal{P}u^P-u^P|_\infty+|\partial\mathcal{P}u^P-\partial u^P|_\infty)$.

Now, for any function φ,

$$|\mathcal{P}\varphi|_{k,\infty}\leq C_k(\mathcal{P})|\varphi|_{k,\infty};$$
$$|\mathcal{P}\varphi-\varphi|_{k,\infty}\leq \varepsilon_k(\mathcal{P})|\varphi|_{\alpha_k(\mathcal{P}),\infty},$$

$\Rightarrow$ need for the study of the blow-up rate $|u^p|_{k,\infty}$.

Main result:

Theorem

Assume $(\mathbf{A}_{b,\sigma}^{k+1})$, $(\mathbf{A}_{f}^{k})$ *and* $(\mathbf{A}_{g}^{0})$, *for some* $1 \leq k \leq +\infty$. *Then, for any* $l = 0 \dots k$, *we have* $u^{p,n}(t,.) \in \mathcal{C}_b^{l+1}$, *for* $p = 0 \dots P$ *and* $n = 0 \dots N-1-l$.

Moreover, for any $l = 0 \dots k$, *the following uniform estimate holds (with positive constants* $C_{P,l}$, α_l *and* β_l*)*

$$\sup_{\substack{t\in[0,t_{N-l}]\\ p=0\dots P}} |u^p(t,.)|_{l,\infty} \leq C_{P,l}(|g|_\infty^{\alpha_l} N^{\beta_l} + 1).$$

NB. Possibly, a better bound holds (current work):

$$\sup_{p=0\dots P} |u^p(t,.)|_{l,\infty} \leq C_{P,l}(|g|_\infty^{\alpha_l}(T-t)^{-\beta_l} + 1).$$

First step: Preliminary estimates.

Lemma

For fixed $\tau > 0$ and a bounded measurable function $\phi(\tau, .)$, set

$$\varphi(t, \tau; x) := \mathbb{E}[\phi(\tau, X_\tau^{t,x})],$$

for $0 \leq t \leq \tau \leq T$ and $x \in \mathbb{R}^d$. Assume $(\mathbf{A}_{b,\sigma}^k)$ for some $k \geq 0$. Then, $\varphi(t, \tau; .) \in \mathcal{C}_b^k$ and, for all $t < \tau$,

$$|\varphi(t, \tau; .)|_{k,\infty} \leq C_k |\phi(\tau, .)|_\infty (\tau - t)^{-\frac{k}{2}}.$$

More generally, if $\phi(\tau, .) \in \mathcal{C}_b^l$ for some $0 \leq l \leq k$, then

$$|\varphi(t, \tau; .)|_{k,\infty} \leq C_k |\phi(\tau, .)|_{l,\infty} (\tau - t)^{-\frac{k-l}{2}}.$$

Now, for fixed $\tau > 0$ and a bounded measurable function $\phi(\tau, .)$, we define the Picard sequence of functions $(\varphi^p(., \tau; .))_{p\geq 0}$ by:

$$\begin{aligned}
\varphi^0(t,\tau;x) &:= \mathbb{E}[\phi(\tau, X_\tau^{t,x})];\\
\varphi^{p+1}(t,\tau;x) &:= \mathbb{E}[\phi(\tau, X_\tau^{t,x})]\\
&\quad + \mathbb{E}[\int_t^\tau f(s, X_s^{t,x}, \varphi^p(s,\tau;X_s^{t,x}),\\
&\qquad \partial\varphi^p(s,\tau;X_s^{t,x})\sigma(s,X_s^{t,x}))\mathrm{d}s].
\end{aligned}$$

Lemma

If $(\mathbf{A}^{k+1}_{b,\sigma})$ *and* $(\mathbf{A}^{k}_{f})$ *hold for some* $k \geq 0$, *and if* $\phi(\tau,.) \in \mathcal{C}^k_b$, *then* $\varphi^p(t,\tau;.) \in \mathcal{C}^{k+1}_b$ *for* $p \geq 0$ *and* $t < \tau$, *and*

$$\begin{aligned} |\varphi^p(t,\tau;.)|_{k,\infty} &\leq C_{p,k}(|\phi(\tau,.)|_{k,\infty}(1+\sqrt{\tau-t}) \\ &+ (|\phi(\tau,.)|^{\alpha_{p,k}}_{k,\infty}+1)(\tau-t)); \\ |\varphi^p(t,\tau;.)|_{k+1,\infty} &\leq C_{p,k+1}(|\phi(\tau,.)|_{k,\infty}((\tau-t)^{-1/2}+1) \\ &+ (|\phi(\tau,.)|^{\alpha_{p,k}}_{k,\infty}+1)\sqrt{\tau-t}), \end{aligned}$$

for some positive constants $C_{p,k}$, $C_{p,k+1}$ *and* $\alpha_{k,p}$.

NB. In particular, the sequence $(\varphi^p(.,\tau;.))_{p\geq 0}$ is well-defined.

Proof.: by induction on p.
We set

$$\tilde{\phi}^p_\tau(s,.) := f(s,.,\varphi^p(s,\tau;.),\partial\varphi^p(s,\tau;.)\sigma(s,.));$$
$$\tilde{\varphi}^p_\tau(t,s;x) := \mathbb{E}\tilde{\phi}^p_\tau(s,X^{t,x}_s),$$

so that $\varphi^{p+1}(t,\tau;.) = \varphi^0(t,\tau;.) + \int_t^\tau \tilde{\varphi}^p_\tau(t,s;.)\mathrm{d}s$.
(i) $\underline{\mathbf{k=0}}$: We assume $(\mathbf{A}^1_{b,\sigma})$, $(\mathbf{A}^0_f)$ and ϕ bounded.

$$\begin{aligned}|\tilde{\varphi}^p_\tau(t,s;.)|_\infty &\leq |\tilde{\phi}^p_\tau(s,.)|_\infty \\ &\leq_c 1 + C_{p,1}(|\phi(\tau,.)|_\infty((\tau-s)^{-1/2}+1) \\ &+ (|\phi(\tau,.)|_\infty^{\alpha_{p,0}}+1)\sqrt{\tau-s}) \\ &\in \mathbf{L}^1((t,\tau),\mathrm{d}s).\end{aligned}$$

$$
\begin{aligned}
|\partial\tilde{\varphi}^p_\tau(t,s;.)|_\infty &\le_c |\tilde{\phi}^p_\tau(s,.)|_\infty (s-t)^{-1/2} \\
&\le_c |\phi(\tau,.)|_\infty ((\tau-s)^{-1/2}+1)(s-t)^{-1/2} \\
&+ (|\phi(\tau,.)|_\infty^{\alpha_{p,0}}+1)\sqrt{\tau-s}(s-t)^{-1/2} \\
&\in \mathbf{L}^1((t,\tau),\mathrm{d}s).
\end{aligned}
$$

(ii) $\underline{\mathbf{k}>\mathbf{0}}$: similarly, using

$$
\begin{aligned}
\partial^k\tilde{\phi}^p_\tau(s,.) &= \partial^k\{f(s,.,\varphi^p(s,\tau;.),\partial\varphi^p(s,\tau;.)\sigma(s,.))\} \\
&= \partial_z f(\theta(s,.))\sigma\partial^{k+1}\varphi^p(s,\tau;.) \\
&+ Q_k\left[(\partial^i f(\theta(s,.)),\partial^j\sigma(s,.),\partial^l\varphi^p(s,\tau;.))_{\substack{i=0\dots k\\ j=0\dots k\\ l=0\dots k}}\right],
\end{aligned}
$$

where $\theta(s,.)$ denotes $(s,.,\varphi(s,\tau;.),\partial\varphi(s,\tau;.)\sigma(s,.))$, and $Q_k[.]$ is a polynomial function.

Second step: Equivalence between the deterministic and the BSDE formulations.

Lemma

Assume $(\mathbf{A}^2_{b,\sigma})$ *and* $(\mathbf{A}^1_f)$. *Then, for all* $p = 0 \dots P$ *and* $n = 0 \dots N-1$, $\mathrm{d}\mathbb{P} \otimes \mathrm{d}t - a.s.$,

$$Y_t^{p,n} = u^{p,n}(t, X_t);$$

$$Z_t^{p,n} = \partial u^{p,n}(t, X_t)\sigma(t, X_t).$$

Proof. For $\varepsilon > 0$, we define the localized driver

$$f^{\varepsilon}(s, x, y, z) := f(s, x, y, z)\mathbb{1}_{s \le \tau - \varepsilon},$$

and $\phi^{\varepsilon}(\tau, .) \in \mathcal{C}^1_b$ s.t. $\mathbb{E}|\phi^{\varepsilon}(\tau, X_\tau) - \phi(\tau, X_\tau)|^2 \to 0$ when $\varepsilon \to 0$.

We show that

- for fixed $\varepsilon > 0$, $Y_t^{p+1,\varepsilon} = \varphi^{p+1,\varepsilon}(t,\tau;X_t)$ and $Z_t^{p+1,\varepsilon} = \partial\varphi^{p+1,\varepsilon}(t,\tau;X_t)\sigma(t,X_t)$.
- $\mathbb{E}\int_0^\tau |Y_t^{p,\varepsilon} - Y_t^p|^2 + |Z_t^{p,\varepsilon} - Z_t^p|^2 \mathrm{d}t \to 0$ when $\varepsilon \to 0$.
- $$\mathbb{E}\int_0^\tau |\varphi^{p+1,\varepsilon}(t,\tau;X_t) - \varphi^{p+1}(t,\tau;X_t)|^2\mathrm{d}t + \mathbb{E}\int_0^\tau |\partial\varphi^{p+1,\varepsilon}(t,\tau;X_t) - \partial\varphi^{p+1}(t,\tau;X_t)|^2\mathrm{d}t \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

The three steps yield the lemma.

Third step: proof of the theorem

Combining the two first steps, we use differentiations of the discretized BSDE.

(i) $\mathbf{k = 1}$.

$$\partial Y_{\cdot}^{p,n} = \partial Y_{t_{n+1}}^{P,n+1} + \int_{\cdot}^{t_{n+1}} \Phi_1^{p-1,n}(\omega, s, \partial Y_s^{p-1,n}, \partial Z_s^{p-1,n})\mathrm{d}s$$
$$- \int_{\cdot}^{t_{n+1}} \partial Z_s^{p,n}\mathrm{d}W_s,$$

with

$$\Phi_1^{p-1,n}(\omega, s, \tilde{y}, \tilde{z}) = \partial_x f(\Theta_s^{p-1,n})\partial X_s + \partial_y f(\Theta_s^{p-1,n})\tilde{y} + \partial_z f(\Theta_s^{p-1,n})\tilde{z}.$$

Then, by the stability property of the time discretization of BSDE (first part of the talk),

$$|\partial Y_t^p| \le C_P((\mathbb{E}|\partial Y_{t_{N-1}}^{P,N-1}|^2)^{1/2} + \sup_p(\mathbb{E}[(\int_0^{t_{N-1}} |\Phi_1^p(s,0,0)|\mathrm{d}s)^2])^{1/2})$$
$$\le C_P(|\partial u^{P,N-1}|_\infty + 1).$$

(ii) $\underline{\mathbf{k \geq 1}}$: By induction on k, and

$$\Phi_k^p(\omega, s, \partial^k Y_s^p, \partial^k Z_s^p) = \partial_z f(\Theta_s^p)\partial^k Z_s^p + \left(A_{0,k}^p + A_{1,k}^p \partial Y_s^p + A_{2,k}^p \partial Z_s^p\right) \partial^{k-1} Z_s^p + R_k^p(s),$$

where

- $A_{0,k}^p, A_{1,k}^p, A_{2,k}^p$ are polynomial functions of $(\partial^i f(\Theta_s^p), \partial^i \sigma(s, X_s), \partial^i X_s, \partial^i (\partial X_s)^{-1})$ for $i = 0 \ldots k$,
- and R_k^p is a polynomial function of the same quantities together with $(\partial^j Y_s^p)$ for $j = 0 \ldots k-1$.

$\longrightarrow$ additional nonlinear terms have to be handled...

A numerical example.

$T = 1, X_0 = 0, b(t,x) = 0, \sigma(t,x) = 1, g(x) = \mathbb{1}_{[K,+\infty)}, K = 0, f(t,x,y,z) = \gamma z, \gamma = 1.$

N	N^X	$N^{\mathcal{P}}$	$\overline{Y}_0$	CPU time	Rel. error
10	1	3	0.70	30 s	16%
10	1	4	0.83	1 mn	1%
10	1	5	0.78	1.5 mn	7%
10	1	6	0.86	1.5 mn	3%
50	1	6	0.85	8 mn	1%

Table: BSDE with indicator terminal function ($Y_0 = 0.841$).

- small error
- but the variance of the error increases with the number of nodes

$\Rightarrow$ need for a careful tuning of the parameters...